

The Flag Major Index and Group Actions on Polynomial Rings

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Abstract

A new extension of the major index, defined in terms of Coxeter elements, is introduced. For the classical Weyl groups of type B , it is equidistributed with length. For more general wreath products it appears in an explicit formula for the Hilbert series of the (diagonal action) invariant algebra.

1 Introduction

1.1 Outline

The *major index*, $\text{major}(\pi)$, of a permutation π in the symmetric group S_n is the sum (possibly zero) of all indices $1 \leq i < n$ for which $\pi(i) > \pi(i+1)$. The *length* of a permutation π is the minimal number of factors in an expression of π as a product of the Coxeter generators $(i, i+1)$, $1 \leq i < n$. A fundamental property of the major index is its equidistribution with the length function [MM]; namely, the number of elements in S_n of a given length k is equal to the number of elements having major index k . Bijective proofs and generalizations were given in [F, FS, Ca, GG, Go, Ro3].

Candidates for a major index for the classical Weyl groups of type B have been suggested by Clarke-Foata [CF1–3], Reiner [Rei1–2], Steingrimsson

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[Ste], and others. Unfortunately, unlike the case of the symmetric group, the various alternatives are not equidistributed with the length function (defined with respect to the Coxeter generators of B_n).

In this paper we present a new definition of the major index for the groups B_n , and more generally for wreath products of the form $C_m \wr S_n$, where C_m is the cyclic group of order m . This major index is shown to be equidistributed with the length function for the groups of type B (the case $m = 2$), and to play a crucial role in the study of the actions of these groups on polynomial rings.

The rest of the paper is organized as follows. The definition of the flag major index is presented in Subsection 1.2 (and, in more detail, in Section 2). Main results are surveyed in Subsection 1.3. Basic properties and equidistribution with length are proved in Section 2. In Section 3 we give a combinatorial interpretation of the flag major index. In Section 4 we study $C_m \wr S_n$ -actions on tensor powers of polynomial rings. The combinatorial interpretation of the flag major index is then applied to obtain a simple representation of the Hilbert Series of the diagonal action invariant algebra.

1.2 The Flag Major Index

The groups $C_m \wr S_n$ are generated by $n - 1$ involutions, s_1, \dots, s_{n-1} , which satisfy the usual Moore-Coxeter relations of S_n , together with an exceptional generator, s_0 , of order m . We consider a different set of generators:

$$t_i := \prod_{j=0}^i s_{i-j} \quad (0 \leq i \leq n-1).$$

These are Coxeter elements in a distinguished flag of parabolic subgroups. See Section 2 for more details.

An element $\pi \in C_m \wr S_n$ has a unique representation as a product

$$\pi = t_{n-1}^{k_{n-1}} t_{n-2}^{k_{n-2}} \cdots t_1^{k_1} t_0^{k_0}$$

with $0 \leq k_i < m(i+1)$ ($\forall i$). Define the *flag major index* of π by

$$\text{flag-major}(\pi) := \sum_{i=0}^{n-1} k_i.$$

For $m = 1$, this definition gives a new interpretation of a well-known parameter.

Claim 1. *For $m = 1$ (i.e., for the symmetric group S_n) the flag major index coincides with the major index.*

See Claim 2.1 below.

1.3 Main Results

From Claim 1 together with MacMahon's classical result it follows that the flag major index is equidistributed with length, for $m = 1$. This property extends to $m = 2$.

Theorem 2. *For $m = 2$ (i.e., for the hyperoctahedral group B_n) the flag major index is equidistributed with length.*

Here “length” is used in the usual sense, in terms of the Coxeter generators s_0, s_1, \dots, s_{n-1} . See Theorem 2.2 below.

For $m \geq 3$, the flag major index is no longer equidistributed with length (with respect to s_0, \dots, s_{n-1}), but nevertheless it does play a central role in the study of naturally defined algebras of polynomials.

Let $P_n := \mathcal{C}[x_1, \dots, x_n]$ be the algebra of polynomials in n indeterminates. There is a natural action of $G := C_m \wr S_n$ on P_n (presented explicitly in Section 4). Consider now the tensor power $P_n^{\otimes t}$ with the natural *tensor action* φ_T of $G^t := G \times \dots \times G$ (t factors), and the corresponding *diagonal action* of G . For more details see Section 4.

The *tensor invariant algebra* TIA is a subalgebra of the *diagonal invariant algebra* DIA. Let $F_D(\bar{q})$, where $\bar{q} = (q_1, \dots, q_t)$, be the multi-variate generating function (Hilbert series) for the dimensions of the homogeneous components in DIA:

$$F_D(\bar{q}) := \sum_{n_1, \dots, n_t \in \mathbf{N}} (\dim_{\mathcal{C}} \text{DIA}_{n_1, \dots, n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

where $\text{DIA}_{n_1, \dots, n_t}$ is the homogeneous piece of multi-degree (n_1, \dots, n_t) in DIA. Define similarly $F_T(\bar{q})$ for TIA. Then

Theorem 3. *For all $m, n, t \geq 1$,*

$$\frac{F_D(\bar{q})}{F_T(\bar{q})} = \sum_{\pi_1 \cdots \pi_t = 1} \prod_{i=1}^t q_i^{\text{flag-major}(\pi_i)}$$

where the sum extends over all t -tuples (π_1, \dots, π_t) of elements in $G = C_m \wr S_n$ such that the product $\pi_1 \pi_2 \cdots \pi_t$ is equal to the identity element.

See Theorem 4.1 below.

2 Flag Major Index and Length

The groups $C_m \wr S_n$ are generated by $n - 1$ involutions, s_1, \dots, s_{n-1} , together with an exceptional generator, s_0 , of order m . The $n - 1$ involutions satisfy the usual Moore-Coxeter relations of S_n ,

$$(s_i s_{i+1})^3 = 1 \quad (1 \leq i < n),$$

$$(s_i s_j)^2 = 1 \quad (|i - j| > 1),$$

while the exceptional generator satisfies the relations:

$$(s_0 s_1)^{2m} = 1,$$

$$s_0 s_i = s_i s_0 \quad (1 < i < n).$$

For $m = 1$, s_0 is the identity element.

Consider now a different set of generators:

$$t_i := \prod_{j=0}^i s_{i-j} \quad (0 \leq i \leq n - 1).$$

These are Coxeter elements [Hu, §3.16] in a distinguished flag of parabolic subgroups

$$1 < G_1 < \dots < G_n = C_m \wr S_n$$

where $G_i \cong C_m \wr S_i$ is the subgroup of $C_m \wr S_n$ generated by s_0, s_1, \dots, s_{i-1} .

An element $\pi \in C_m \wr S_n$ has a unique representation as a product

$$\pi = t_{n-1}^{k_{n-1}} t_{n-2}^{k_{n-2}} \cdots t_1^{k_1} t_0^{k_0}$$

with $0 \leq k_i < m(i + 1)$ ($\forall i$). Define the *flag major index* of π by

$$\text{flag-major}(\pi) := \sum_{i=0}^{n-1} k_i.$$

For $m = 1$, this definition gives a new interpretation of a well-known parameter.

Claim 2.1. *For $m = 1$ (i.e., for the symmetric group S_n) the flag major index coincides with the major index.*

Proof. Consider the natural action of S_n on the letters $1, \dots, n$, where s_i ($1 \leq i < n$) acts as the transposition $(i, i+1)$. Here $t_0 = s_0$ is the identity permutation, whereas for $1 \leq r < n$, $t_r = s_r \cdots s_1$ acts as the cycle $(r+1, r, \dots, 2, 1)$. The claim may be proved by induction on the exponents k_i . Obviously, equality holds when $k_1 = \dots = k_{n-1} = 0$. It suffices to prove that for any $\pi = t_r^{k_r} \cdots t_1^{k_1}$ with $1 \leq r < n$ and $0 \leq k_r < r$, $\text{major}(t_r \pi) - \text{major}(\pi) = 1$. In this case $\pi(j) = j$ for $j > r+1$ and $\pi(r+1) \in \{2, 3, \dots, r+1\}$. Observe that, for $1 \leq j \leq r+1$, $t_r \pi(j) > t_r \pi(j+1)$ if and only if $\pi(j) > \pi(j+1)$, unless $\pi(j) = 1$ or $\pi(j+1) = 1$. If $\pi(j) = 1$ (this holds for a unique $1 \leq j \leq r$) then $t_r \pi(j) = r+1$, so that $\pi(j-1) > \pi(j) < \pi(j+1)$ whereas $t_r \pi(j-1) < t_r \pi(j) > t_r \pi(j+1)$; thus the descent at $j-1$ in π is replaced by a descent at j in $t_r \pi$. The special case $j = 1$ is similar. \square

In particular, by MacMahon's classical result, the flag major index is equidistributed with length (for $m = 1$). This property extends to $m = 2$.

Theorem 2.2. *For $m = 2$ (i.e., for the hyperoctahedral group B_n), the flag major index is equidistributed with length.*

Here “length” is used in the usual sense, in terms of the Coxeter generators s_0, s_1, \dots, s_{n-1} . Theorem 2.2 can be proved by an explicit bijection.

Proof. For $0 \leq m < 2n$ define $r_{n,m} \in B_n$ by

$$r_{n,m} := \begin{cases} id, & \text{if } m = 0; \\ \prod_{j=n-m}^{n-1} s_j, & \text{if } 0 < m \leq n; \\ \prod_{j=0}^{m-n-1} s_{m-n-j} \prod_{j=0}^{n-1} s_j, & \text{if } n < m < 2n, \end{cases}$$

where id is the identity element in B_n .

Note that the length of $r_{n,m}$ is m . The set $\{r_{n,m} | 0 \leq m < 2n\}$ forms a complete set of representatives of minimal length for the left cosets of B_{n-1} in B_n . It follows that every element $\pi \in B_n$ has a unique representation as a product $\pi = \prod_{i=1}^n r_{n+1-i, m_{n+1-i}}$, where $0 \leq m_j < 2j$ for every j , and then $\ell(\pi) = \sum_{j=1}^n m_j$.

On the other hand, every element in B_n has a unique representation as a product of the form $\prod_{i=1}^n t_{n-i}^{k_{n-i}}$, where $0 \leq k_j < 2(j+1)$ for every j . By definition, $\text{flag-major}(\pi) = \sum_{j=1}^n k_{j-1}$.

It follows that the map $\phi : B_n \longrightarrow B_n$ defined by

$$\phi\left(\prod_{i=1}^n r_{n+1-i, m_{n+1-i}}\right) := \prod_{i=1}^n t_{n-i}^{m_{n+1-i}}$$

is bijective and sends the length function to the flag major index. \square

For $m \geq 3$ (when $C_m \wr S_n$ is no longer a Coxeter group), the flag major index is no longer equidistributed with length (with respect to s_0, \dots, s_{n-1}); nevertheless, it does play a central role in the study of naturally defined algebras of polynomials, as will be shown in Section 4.

3 A Combinatorial Interpretation

The standard major index for permutations has a natural generalization to any finite sequence of letters from a linearly ordered alphabet (see, e.g., [F]); namely, for a finite sequence $a = (a_1, a_2, \dots, a_n)$ of letters from a linearly ordered alphabet, define $\text{major}(a)$ to be the sum (possibly zero) of all indices $1 \leq i < n$ for which $a_i > a_{i+1}$.

Let $\omega \in \mathbf{C}$ be a primitive m -th root of unity. An element of the wreath product $C_m \wr S_n$ (where C_m is the cyclic group of order m) may be described as a *generalized permutation* $\pi = (\pi(1), \pi(2), \dots, \pi(n))$, where, for every i , $\pi(i) \in \mathbf{C}$, $\frac{\pi(i)}{|\pi(i)|}$ is a power of ω , and the sequence of absolute values $|\pi| := (|\pi(1)|, |\pi(2)|, \dots, |\pi(n)|)$ is a permutation in S_n . In this setting, the involutions s_i ($1 \leq i < n$) are defined by

$$s_i(j) := \begin{cases} i+1, & \text{if } j = i; \\ i, & \text{if } j = i+1; \\ j, & \text{otherwise,} \end{cases}$$

whereas the exceptional generator s_0 is defined by

$$s_0(j) := \begin{cases} \omega \cdot 1, & \text{if } j = 1; \\ j, & \text{otherwise.} \end{cases}$$

In particular, $C_2 \wr S_n$ is the group of *signed permutations*, also known as the hyperoctahedral group, or the classical Weyl group of type B .

Consider now the linearly ordered alphabet

$$1 \cdot \omega^{m-1} < \dots < n \cdot \omega^{m-1} < \dots < 1 \cdot \omega^1 < \dots < n \cdot \omega^1 < 1 \cdot \omega^0 < \dots < n \cdot \omega^0.$$

In this section we prove

Theorem 3.1. *For any $\pi \in C_m \wr S_n$,*

$$(3.1) \quad \text{flag-major}(\pi) = m \cdot \text{major}(\pi) + \sum_{j=0}^{m-1} j \cdot \# \left\{ i : \frac{\pi(i)}{|\pi(i)|} = \omega^j \right\},$$

where $\text{major}(\pi)$ is defined with respect to the above order.

To simplify the proof some notations are needed. For any generalized permutation $\pi \in C_m \wr S_n$ and $1 \leq i \leq n$ define

$$(3.2) \quad \text{Log}_\omega \pi(i) := \min \left\{ d \geq 0 : \omega^d = \frac{\pi(i)}{|\pi(i)|} \right\}.$$

Then clearly

$$(3.3) \quad \sum_{i=1}^n \text{Log}_\omega \pi(i) = \sum_{j=0}^{m-1} j \cdot \# \left\{ i : \frac{\pi(i)}{|\pi(i)|} = \omega^j \right\}.$$

Denote the right hand side of (3.1) by $\text{major}_{m,n}(\pi)$. By (3.3),

$$(3.4) \quad \text{major}_{m,n}(\pi) = m \cdot \text{major}(\pi) + \sum_{i=1}^n \text{Log}_\omega \pi(i).$$

Lemma 3.2. *For any $\pi \in C_m \wr S_n$ with $\pi(n) \neq \omega^{m-1}$,*

$$\text{major}_{m,n}(t_{n-1}\pi) - \text{major}_{m,n}(\pi) = 1.$$

Proof of Lemma 3.2. By definition,

$$t_{n-1}(i) = \begin{cases} i-1, & \text{if } i \neq 1; \\ \omega \cdot n, & \text{if } i = 1. \end{cases}$$

Hence, for any $\pi \in C_m \wr S_n$,

$$(3.5) \quad t_{n-1}\pi(j) = \begin{cases} \omega^{\text{Log}_\omega \pi(j)} \cdot (\pi(j) - 1), & \text{if } |\pi(j)| \neq 1; \\ \omega^{\text{Log}_\omega \pi(j)+1} \cdot n, & \text{if } |\pi(j)| = 1. \end{cases}$$

Let $1 \leq i_0 \leq n$ be the unique index for which $|\pi(i_0)| = 1$.

Case (a). $\text{Log}_\omega \pi(i_0) < m - 1$.

It follows from (3.5) that if $\text{Log}_\omega \pi(i_0) < m - 1$ then

$$\text{major}(t_{n-1}\pi) = \text{major}(\pi)$$

with respect to the linear order defined before Theorem 3.1. Hence, in this case

$$\begin{aligned} \text{major}_{m,n}(t_{n-1}\pi) &= m \cdot \text{major}(t_{n-1}\pi) + \sum_{i=1}^n \text{Log}_\omega t_{n-1}\pi(i) = \\ &= m \cdot \text{major}(\pi) + \sum_{i=1}^n \text{Log}_\omega t_{n-1}\pi(i) = \\ &= m \cdot \text{major}(\pi) + \sum_{i \neq i_0} \text{Log}_\omega \pi(i) + (\text{Log}_\omega \pi(i_0) + 1) = \text{major}_{m,n}(\pi) + 1. \end{aligned}$$

Case (b). $\text{Log}_\omega \pi(i_0) = m - 1$.

In this case $\pi(i_0) = 1 \cdot \omega^{m-1}$ and $t_{n-1}\pi(i_0) = n$. By assumption $i_0 \neq n$, so that π has a descent at $i_0 - 1$ (unless $i_0 = 1$), whereas $t_{n-1}\pi$ has a descent at i_0 . Thus

$$\text{major}(t_{n-1}\pi) - \text{major}(\pi) = 1$$

and $\text{Log}_\omega t_{n-1}\pi(i_0) = 0$. Hence, in this case,

$$\begin{aligned} \text{major}_{m,n}(t_{n-1}\pi) &= m \cdot \text{major}(t_{n-1}\pi) + \sum_{i=1}^n \text{Log}_\omega t_{n-1}\pi(i) = \\ &= m \cdot (\text{major}(\pi) + 1) + \sum_{i \neq i_0} \text{Log}_\omega \pi(i) = \text{major}_{m,n}(\pi) + m - (m - 1). \end{aligned}$$

□

Proof of Theorem 3.1. By induction on n . Obviously, Theorem 3.1 holds in the group $C_m \wr S_1 = \langle s_0 \rangle$. Assume that the theorem holds in the group $C_m \wr S_n$, for some $n \geq 1$. Any element of $C_m \wr S_{n+1}$ has the form $t_n^{k_n} \pi$, where $\pi \in C_m \wr S_n$ and $0 \leq k_n < m(n+1)$. By definition,

$$\text{flag-major}(t_n^{k_n} \pi) = k_n + \text{flag-major}(\pi)$$

and

$$\text{major}_{m,n+1}(\pi) = \text{major}_{m,n}(\pi).$$

Hence, by the induction hypothesis, it suffices to prove that for any $\pi \in C_m \wr S_n$ and $0 \leq k_n < m(n+1)$

$$\text{major}_{m,n+1}(t_n^{k_n} \pi) - \text{major}_{m,n+1}(\pi) = k_n.$$

This equality readily follows from iterations of Lemma 3.2, thereby completing the proof. \square

4 Diagonal Action on Tensor Powers

Let $P_n := \mathbf{C}[x_1, \dots, x_n]$ be the algebra of polynomials in n indeterminates. There is a natural action of $G := C_m \wr S_n$ on P_n , $\varphi : G \rightarrow \text{Aut}(P_n)$, defined on generators by

$$\begin{aligned} \varphi(s_0)(x_j) &= \begin{cases} \omega \cdot x_j, & \text{if } j = 1; \\ x_j, & \text{otherwise,} \end{cases} \quad (\omega := \exp(2\pi i/m) \in \mathbf{C}) \\ \varphi(s_i)(x_j) &= \begin{cases} x_{i+1}, & \text{if } j = i; \\ x_i, & \text{if } j = i+1; \\ x_j, & \text{otherwise,} \end{cases} \quad (1 \leq i \leq n-1) \end{aligned}$$

where each $\varphi(s_i), 0 \leq i \leq n-1$, is extended to an algebra automorphism of P_n . Equivalently, in terms of generalized permutations,

$$\varphi(\pi)(x_j) = \frac{\pi(j)}{|\pi(j)|} \cdot x_{|\pi(j)|} \quad (\forall \pi \in G, 1 \leq j \leq n)$$

extended multiplicatively to monomials and additively to all of P_n .

Consider now the tensor power $P_n^{\otimes t} := P_n \otimes \dots \otimes P_n$ (t factors) with the natural *tensor action* φ_T of $G^t := G \times \dots \times G$ (t factors). The diagonal embedding

$$d : G \hookrightarrow G^t$$

defined by

$$g \mapsto (g, \dots, g) \in G^t \quad (\forall g \in G)$$

defines the *diagonal action* of G on $P_n^{\otimes t}$:

$$\varphi_D := \varphi_T \circ d.$$

The *tensor invariant algebra*

$$\text{TIA} := \{\bar{p} \in P_n^{\otimes t} \mid \varphi_T(\bar{g})(\bar{p}) = \bar{p}, \forall \bar{g} \in G^t\}$$

is a subalgebra of the *diagonal invariant algebra*

$$\text{DIA} := \{\bar{p} \in P_n^{\otimes t} \mid \varphi_D(g)(\bar{p}) = \bar{p}, \forall g \in G\}.$$

Note that $\text{TIA} = (P_n^G)^{\otimes t}$, where P_n^G is the subalgebra of P_n invariant under $\varphi(G)$.

The algebra $P_n^{\otimes t}$ is \mathbf{N}^t -graded by multi-degree, where $\mathbf{N} := \{0, 1, 2, \dots\}$. Let $F_D(\bar{q})$, where $\bar{q} = (q_1, \dots, q_t)$, be the multivariate generating function (Hilbert series) for the dimensions of the homogeneous components in DIA:

$$F_D(\bar{q}) := \sum_{n_1, \dots, n_t \in \mathbf{N}} (\dim_{\mathcal{C}} \text{DIA}_{n_1, \dots, n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

where $\text{DIA}_{n_1, \dots, n_t}$ is the homogeneous piece of multi-degree (n_1, \dots, n_t) in DIA. Define similarly $F_T(\bar{q})$ for TIA. Then

Theorem 4.1. *For all $m, n, t \geq 1$,*

$$\frac{F_D(\bar{q})}{F_T(\bar{q})} = \sum_{\pi_1 \cdots \pi_t = 1} \prod_{i=1}^t q_i^{\text{flag-major}(\pi_i)}$$

where the sum extends over all t -tuples (π_1, \dots, π_t) of elements in $G = C_m \wr S_n$ such that the product $\pi_1 \pi_2 \cdots \pi_t$ is equal to the identity element.

5 Proof of Theorem 4.1

5.1 Preliminaries

To prove Theorem 4.1 we need two theorems of Garsia and Gessel.

A *t-partite partition with n parts* (in the sense of Gordon [G] and Garsia-Gessel [GG]) is a sequence $f = (f_1, \dots, f_t)$ of non-negative-integer valued functions $f_i : \{1, 2, \dots, n\} \rightarrow \mathbf{N}$, $1 \leq i \leq t$, satisfying the condition:

$$\text{If } f_i(j) = f_i(j+1) \text{ for all } i < i_0, \text{ then } f_{i_0}(j) \geq f_{i_0}(j+1) \quad (\forall i_0, j).$$

In particular, for $i_0 = 1$:

$$f_1(1) \geq f_1(2) \geq \dots \geq f_1(n) \geq 0,$$

i.e., f_1 is a partition with (at most) n parts.

Example. $n = 4, t = 2$; $f_1 = (1, 1, 0, 0)$, $f_2 = (1, 0, 2, 2)$.

Let $B_{t,n}$ be the set of all t -partite partitions $f = (f_1, \dots, f_t)$ with n parts. Denote the sum $\sum_{j=1}^n f_i(j)$ by $|f_i|$. The following theorem was first proved by Garsia and Gessel.

Theorem GG1 [GG, Theorem 2.2 and Remark 2.2]

$$\sum_{f \in B_{t,n}} q_1^{|f_1|} \dots q_t^{|f_t|} = \frac{\sum_{\pi_1 \dots \pi_t = 1} \prod_{i=1}^t q_i^{\text{major}(\pi_i)}}{\prod_{i=1}^t \prod_{j=1}^n (1 - q_i^j)}$$

where the sum in the numerator of the right-hand side extends over all t -tuples (π_1, \dots, π_t) of permutations in S_n such that the product $\pi_1 \pi_2 \dots \pi_t$ is equal to the identity permutation.

Let n_1, \dots, n_r be non-negative integers such that $\sum_{i=1}^r n_i = n$. Recall that the q -multinomial coefficient $\begin{bmatrix} n \\ n_1, \dots, n_r \end{bmatrix}_q$ is defined by:

$$\begin{aligned} [0]!_q &:= 1, \\ [n]!_q &:= [n-1]!_q \cdot (1 + q + \dots + q^{n-1}) \quad (n \geq 1), \\ \begin{bmatrix} n \\ n_1 \dots n_r \end{bmatrix}_q &:= \frac{[n]!_q}{[n_1]!_q \dots [n_r]!_q}. \end{aligned}$$

Now let (N_1, \dots, N_r) be a partition of the set $N := \{1, \dots, n\}$. For each $1 \leq i \leq r$ let π_i be a permutation on the elements of N_i . Recall that a permutation $\sigma \in S_n$ is a *shuffle* of $\pi_1, \pi_2, \dots, \pi_r$ if, for every i , the letters of N_i appear in σ in the same order as the corresponding letters appear in π_i .

Example. $N_1 = \{1, 2, 4\}$, $N_2 = \{3, 5\}$; $\pi_1 = 241$, $\pi_2 = 35$. Here $\sigma = 32541$ is a shuffle of π_1 and π_2 .

Theorem GG2 [GG, Theorem 3.1] Let $\Omega(\pi_1, \dots, \pi_r)$ be the collection of all shuffles of given permutations $\pi_1, \pi_2, \dots, \pi_r$. Then

$$\sum_{\sigma \in \Omega(\pi_1, \dots, \pi_r)} q^{\text{major}(\sigma)} = \begin{bmatrix} n \\ n_1 \dots n_r \end{bmatrix}_q q^{\text{major}(\pi_1) + \dots + \text{major}(\pi_r)},$$

where n_i is the number of elements acted upon by π_i ($1 \leq i \leq r$).

5.2 The Case $m = 1$

Before we get to the actual proof of Theorem 4.1, let us sketch the proof of the case $m = 1$ (i.e., $G = S_n$). This is done as an indication of the structure of the general case; full details will be given in the actual proof.

The tensor invariant algebra TIA is equal to $(P_n^G)^{\otimes t}$, and P_n^G is freely generated (as an algebra) by the n elementary symmetric functions in n indeterminates x_1, \dots, x_n . Thus

$$F_T(\bar{q}) = \frac{1}{\prod_{i=1}^t \prod_{j=1}^n (1 - q_i^j)}.$$

The diagonal invariant algebra DIA is linearly spanned by the polynomials

$$\sum_{g \in S_n} \varphi_D(g)(\bar{x}^f)$$

where \bar{x}^f runs through all the monomials in $P_n^{\otimes t}$. Now $\{\bar{x}^f \mid f \in B_{t,n}\}$ is a complete set of representatives for the orbits of monomials in $P_n^{\otimes t}$ under the diagonal action φ_D of S_n . It follows that a *basis* for DIA is

$$\left\{ \sum_{g \in S_n} \varphi_D(g)(\bar{x}^f) \mid f \in B_{t,n} \right\}.$$

Therefore, the generating function $F_D(\bar{q})$ is given by the left-hand side of the equation in Theorem GG1. By Claim 2.1, this proves Theorem 4.1 for $m = 1$. (Theorem GG2 is not needed in this case.)

In the next subsection we shall construct an explicit basis for the diagonal invariant algebra DIA, for general m . In Subsection 5.5 we shall compute its generating function.

5.3 A Basis for DIA

Consider now the general case of $G = C_m \wr S_n$. A linear basis for $P_n^{\otimes t}$ consists of the (tensor) monomials

$$\bar{x}^f := \bigotimes_{i=1}^t \prod_{j=1}^n x_j^{f_i(j)},$$

where $f_i(j)$ are non-negative integers ($\forall i, j$). Let F denote the set of all such multi-powers f . The canonical projection $\pi : P_n^{\otimes t} \rightarrow \text{DIA}$ is defined by

$$\pi(\bar{p}) := \sum_{g \in G} \varphi_D(g)(\bar{p}) \quad (\forall \bar{p} \in P_n^{\otimes t}),$$

so that

$$\text{DIA} = \text{span } \{\pi(\bar{x}^f) | f \in F\}.$$

We are looking for a subset B of F such that the set $\{\pi(\bar{x}^f) | f \in B\}$ is a basis for DIA.

First, let

$$F_0 := \{f \in F \mid \sum_{i=1}^t f_i(j) \equiv 0 \pmod{m} \ (\forall j)\}.$$

Claim 5.1. *For $f \in F$,*

$$\pi(\bar{x}^f) \neq 0 \iff f \in F_0.$$

Proof.

$$\varphi_D(s_0)(\bar{x}^f) = \omega^{\alpha(f)} \cdot \bar{x}^f,$$

where

$$\alpha(f) := \sum_{i=1}^t f_i(1).$$

Now, for any $\alpha \in \mathbf{Z}$:

$$\sum_{k=0}^{m-1} (\omega^\alpha)^k = \begin{cases} m, & \text{if } \alpha \equiv 0 \pmod{m}; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if C is any left coset in G of the subgroup generated by s_0 , then

$$\sum_{g \in C} \varphi_D(g)(\bar{x}^f) = 0$$

unless

$$\sum_{i=1}^t f_i(1) \equiv 0 \pmod{m}.$$

Replacing s_0 by its $n - 1$ conjugates, it follows that

$$\pi(\bar{x}^f) \neq 0 \implies f \in F_0.$$

For the converse, let H be the normal commutative subgroup of G generated by s_0 and its conjugates: H consists of all “generalized identity permutations” $h \in G$ satisfying $|h(i)| = i \ (\forall i)$. Let \hat{G} be the subgroup of G

consisting of all “unsigned permutations” $\hat{g} \in G$, satisfying $\text{Log}_\omega \hat{g}(i) = 0$ ($\forall i$). Then $\hat{G} \cong S_n$ and G is the semidirect product of \hat{G} and H , hence each $g \in G$ has a unique representation as a product

$$g = \hat{g}h \quad (\hat{g} \in \hat{G}, h \in H).$$

For any $f \in F_0$, $\varphi_D(h)(\bar{x}^f) = \bar{x}^f$ for all $h \in H$. Hence, for $f \in F_0$,

$$\sum_{g \in \hat{g}H} \varphi_D(g)(\bar{x}^f) = |H| \cdot \varphi_D(\hat{g})(\bar{x}^f) \quad (\forall \hat{g} \in \hat{G}).$$

In other words, the sum over a coset of H in G is a monomial multiplied by a positive integer. It follows that

$$f \in F_0 \implies \pi(\bar{x}^f) \neq 0.$$

□

For $\bar{p} = \sum_{f \in F} c_f \bar{x}^f \in P_n^{\otimes t}$ (a finite sum), define the *support*

$$\text{supp}(\bar{p}) := \{f \in F \mid c_f \neq 0\}.$$

The following result is a consequence of the proof of Claim 5.1.

Claim 5.2. *For $f, h \in F_0$, the following are equivalent:*

- (i) $\text{supp}(\pi(\bar{x}^f)) \cap \text{supp}(\pi(\bar{x}^h)) \neq \emptyset$;
- (ii) $\pi(\bar{x}^f) = \pi(\bar{x}^h)$;
- (iii) $\exists \sigma \in S_n$, such that

$$h_i(j) = f_i(\sigma(j)) \quad (\forall i, j).$$

Let us now describe explicitly a subset B of F_0 such that $\{\pi(\bar{x}^f) \mid f \in B\}$ is a basis for DIA. This subset will, of course, be a complete set of representatives for the orbits of all monomials $\{\bar{x}^f \mid f \in F_0\}$ under the action of S_n described in Claim 5.2(iii).

Intuitively, we classify the exponents $f_i(j)$ according to their vector of residues modulo m , and attach a t -partite partition to each possible vector of residues. Formally, define

$$R := \{(r_1, \dots, r_t) \in \mathbf{Z}^t \mid 0 \leq r_i < m \ (\forall i) \text{ and } \sum_{i=1}^t r_i \equiv 0 \pmod{m}\}$$

and choose an arbitrary linear order \leq_R on R .

Define a bijection

$$\theta : F_0 \longrightarrow F \times R^n$$

by

$$\theta(f) := (h, r),$$

where $h = (h_i(j))_{i,j} \in F$ and $r = (r_i(j))_{i,j} \in R^n$ are defined as the quotients and remainders, respectively, obtained when the entries of $f = (f_i(j))_{i,j} \in F_0$ are divided by m :

$$f_i(j) = m \cdot h_i(j) + r_i(j) \quad (\forall i, j).$$

Now let B be the set of all $f \in F_0$ such that $\theta(f) = (h, r)$ satisfies:

- (i) $r_*(1) \geq_R \dots \geq_R r_*(n)$, where $r_*(j) := (r_1(j), \dots, r_t(j)) \in R$ ($\forall j$);
- (ii) If $r_*(j) = r_*(j+1)$, and also $h_i(j) = h_i(j+1)$ for all $i < i_0$, then

$$h_{i_0}(j) \geq h_{i_0}(j+1).$$

We can interpret $f \in B$ as follows: For each $r_* \in R$ there is a range (possibly empty) of indices $j_1 < j \leq j_2$ for which $r_*(j) = r_*$, and a sequence of vectors $(h_*(j))_{j=j_1+1}^{j_2}$ which forms a t -partite partition with $j_2 - j_1$ parts (as defined in Subsection 5.1 above). The total number of vectors (for all $r_* \in R$) is n .

Clearly, B is a complete system of representatives for the orbits of F_0 under the action of S_n defined in Claim 5.2(iii). By Claims 5.1 and 5.2 we conclude

Lemma 5.3. *The set*

$$\{\pi(\bar{x}^f) \mid f \in B\}$$

is a homogeneous basis for DIA.

Corollary 5.4. *The generating function for DIA is*

$$F_D(\bar{q}) = \sum_{f \in B} q_1^{|f_1|} \dots q_t^{|f_t|} = \sum_{(n_r)} \left[\prod_{r \in R} (q_1^{r_1} \dots q_t^{r_t})^{n_r} F_{t, n_r}(q_1^m, \dots, q_t^m) \right],$$

where $F_{t,n}(q_1, \dots, q_t)$ is the generating function for t -partite partitions described in Theorem GG1, and the sum is over all partitions of n into non-negative integers (n_r) indexed by $r \in R$.

Example. $m = t = 2$.

In this case $R = \{(1, 1), (0, 0)\}$, so that $f = (f_1(j), f_2(j))_{1 \leq j \leq n}$ belongs to B if and only if there exists an integer $0 \leq k \leq n$ such that:

- (i) For $1 \leq j \leq k$, $f_1(j)$ and $f_2(j)$ are both odd (i.e., $r_*(j) = (1, 1)$); and for $k < j \leq n$ they are both even (i.e., $r_*(j) = (0, 0)$).
- (ii) The vector pairs $(h_1(j), h_2(j))_{1 \leq j \leq k}$ and $(h_1(j), h_2(j))_{k < j \leq n}$ are both 2-partite partitions.

5.4 Constructing Sequences of Generalized Permutations

In this subsection we construct a bijection between t -tuples of generalized permutations in G with product equal to the identity element, and data consisting of numbers, sets and permutations defined below. This bijection will allow us to apply Theorem GG2 in the computation of the generating function, to be carried out in Subsection 5.5.

Every sequence $(\pi_1, \dots, \pi_t) \in G^t$ such that $\pi_t \cdots \pi_1 = 1$ may be constructed using the following steps:

Step 1. Choose non-negative integers $(n_r)_{r \in R}$ such that

$$\sum_{r \in R} n_r = n.$$

Step 2. For each $1 \leq i \leq t$ choose a set-partition $(N_r^{(i)})_{r \in R}$ of $N := \{1, \dots, n\}$ such that

$$\#N_r^{(i)} = n_r \quad (\forall r \in R),$$

where $\#S$ denotes the size of the set S .

Step 3. For each $1 \leq i \leq t - 1$ and $r \in R$ choose a bijective function (“permutation”) $\hat{\pi}_r^{(i)} : N_r^{(i)} \rightarrow N_r^{(i+1)}$ (essentially, $\hat{\pi}_r^{(i)} \in S_{n_r}$); and define $\hat{\pi}_r^{(t)} : N_r^{(t)} \rightarrow N_r^{(1)}$ such that $\hat{\pi}_r^{(t)} \cdots \hat{\pi}_r^{(1)} = 1$.

Step 4. For each $1 \leq i \leq t$ define a permutation $\hat{\pi}_i \in S_n$ and a generalized permutation $\pi_i \in G$ as follows: if $1 \leq j \leq n$ and $j \in N_r^{(i)}$ then

$$\hat{\pi}_i(x_j) := \hat{\pi}_r^{(i)}(x_j)$$

and

$$\pi_i(x_j) := \omega^{r_i} \cdot \hat{\pi}_r^{(i)}(x_j).$$

Conversely, let $(\pi_1, \dots, \pi_t) \in G^t$ satisfy $\pi_t \cdots \pi_1 = 1$. For each $1 \leq j \leq n$ and $1 \leq i \leq t$ let

$$r_i(j) := \text{Log}_\omega \pi_i(|\pi_{i-1} \cdots \pi_1(j)|).$$

Then

$$r_*(j) := (r_1(j), \dots, r_t(j)) \in R,$$

since $\pi_t \cdots \pi_1 = 1$.

Now define $(n_r)_{r \in R}$ and $(N_r^{(i)})_{r \in R}$ by:

$$N_r^{(1)} := \{1 \leq j \leq n \mid r_*(j) = r\} \quad (\forall r \in R),$$

$$n_r := \#N_r^{(1)} \quad (\forall r \in R),$$

$$N_r^{(i)} := \{|\pi_{i-1} \cdots \pi_1(j)| : j \in N_r^{(1)}\} \quad (\forall r \in R, 2 \leq i \leq t),$$

and

$$\pi_r^{(i)} := \pi_i|_{N_r^{(i)}} \quad (\forall r \in R, 1 \leq i \leq t).$$

To sum up, there is a bijection

$$(\pi_1, \dots, \pi_t) \leftrightarrow ((n_r), (N_r^{(i)}), (\hat{\pi}_r^{(i)}))$$

between t -tuples of generalized permutations in G with product equal to the identity element, and data consisting of numbers, sets and permutations as above.

5.5 Computation of the Generating Function

In this subsection we prove the claim of Theorem 4.1, namely that

$$(5.1) \quad \frac{F_D(\bar{q})}{F_T(\bar{q})} = \sum_{\substack{\pi_1, \dots, \pi_t \in G \\ \pi_t \cdots \pi_1 = 1}} \prod_{i=1}^t q_i^{\text{flag-major}(\pi_i)}.$$

From Theorem 3.1 and from the bijection in the previous subsection it follows that the right-hand side of (5.1) is equal to

$$\begin{aligned}
(5.2) \quad \text{RHS} &= \sum_{\substack{\pi_1, \dots, \pi_t \in G \\ \pi_t \cdots \pi_1 = 1}} \prod_{i=1}^t q_i^{\text{flag-major}(\pi_i)} = \\
&= \sum_{(n_r)} \sum_{(N_r^{(i)})} \sum_{(\hat{\pi}_r^{(i)})} \prod_{i=1}^t q_i^{m \cdot \text{major}(\pi_i) + \sum_{r \in R} n_r r_i} = \\
&= \sum_{(n_r)} \left[\prod_{r \in R} (q_1^{r_1} \cdots q_t^{r_t})^{n_r} \sum_{(N_r^{(i)})} \sum_{(\hat{\pi}_r^{(i)})} \prod_{i=1}^t q_i^{m \cdot \text{major}(\pi_i)} \right].
\end{aligned}$$

Now, $\text{TIA} = (P_n^G)^{\otimes t}$, and P_n^G consists of all the symmetric polynomials in x_1^m, \dots, x_n^m . Thus

$$(5.3) \quad F_T(\bar{q}) = \frac{1}{\prod_{i=1}^t \prod_{j=1}^n (1 - q_i^{mj})},$$

Combining (5.3) with Corollary 5.4 (from the end of Subsection 5.3), we obtain that the left-hand side of (5.1) is equal to

$$(5.4) \quad \text{LHS} = \frac{F_D(\bar{q})}{F_T(\bar{q})} = \prod_{i=1}^t \prod_{j=1}^n (1 - q_i^{mj}) \cdot \sum_{(n_r)} \left[\prod_{r \in R} (q_1^{r_1} \cdots q_t^{r_t})^{n_r} F_{t, n_r}(q_1^m, \dots, q_t^m) \right].$$

Comparing (5.2) with (5.4) we conclude that it suffices to show that, for every choice of $(n_r)_{r \in R}$,

$$(5.5) \quad \prod_{i=1}^t \prod_{j=1}^n (1 - q_i^{mj}) \cdot \prod_{r \in R} F_{t, n_r}(q_1^m, \dots, q_t^m) = \sum_{(N_r^{(i)})} \sum_{(\hat{\pi}_r^{(i)})} \prod_{i=1}^t q_i^{m \cdot \text{major}(\pi_i)}.$$

Theorem GG1 above gives an explicit expression for F_{t, n_r} :

$$F_{t, n_r}(q_1^m, \dots, q_t^m) = \left[\prod_{i=1}^t \prod_{j=1}^{n_r} (1 - q_i^{mj}) \right]^{-1} \cdot \sum_{\hat{\pi}_r^{(t)} \dots \hat{\pi}_r^{(1)} = 1} \prod_{i=1}^t q_i^{m \cdot \text{major}(\hat{\pi}_r^{(i)})},$$

where $\hat{\pi}_r^{(i)} \in S_{n_r}$ are unsigned permutations.

Denoting $q := q_i^m$, the definition of q -multinomial coefficients gives

$$\prod_{j=1}^n (1 - q^j) \cdot \left[\prod_{r \in R} \prod_{j=1}^{n_r} (1 - q^j) \right]^{-1} = \left[\begin{matrix} n \\ (n_r)_{r \in R} \end{matrix} \right]_q,$$

so that the left-hand side of (5.5) is equal to

$$\prod_{i=1}^t \left[\begin{matrix} n \\ (n_r)_{r \in R} \end{matrix} \right]_{q_i^m} \cdot \sum_{(\hat{\pi}_r^{(i)})_{r \in R}} \prod_{r \in R} \prod_{i=1}^t q_i^{m \cdot \text{major}(\hat{\pi}_r^{(i)})}.$$

Here the sum is over all choices of $\hat{\pi}_r^{(i)} \in S_{n_r}$ ($r \in R$, $1 \leq i \leq t$) such that $\hat{\pi}_r^{(t)} \dots \hat{\pi}_r^{(1)} = 1$ ($\forall r$).

Thus, all we need to prove is that, for every choice of (n_r) and $(\hat{\pi}_r^{(i)})$,

$$(5.6) \quad \prod_{i=1}^t \left[\begin{matrix} n \\ (n_r) \end{matrix} \right]_{q_i^m} \cdot \prod_{r \in R} \prod_{i=1}^t q_i^{m \cdot \text{major}(\hat{\pi}_r^{(i)})} = \sum_{(N_r^{(i)})_{r \in R}} \prod_{i=1}^t q_i^{m \cdot \text{major}(\pi_i)}.$$

Since the choice of the partition $(N_r^{(i)})_{r \in R}$ of N can be made independently for each value of i , we can “interchange” the sum and product in the right-hand side of (5.6), and thus it suffices to show that (again, denoting $q := q_i^m$):

$$(5.7) \quad \left[\begin{matrix} n \\ (n_r) \end{matrix} \right]_q \cdot \prod_{r \in R} q^{\text{major}(\hat{\pi}_r^{(i)})} = \sum_{(N_r^{(i)})_{r \in R}} q^{\text{major}(\pi_i)} \quad (\forall i, (n_r), (\hat{\pi}_r^{(i)})).$$

Now this amounts exactly to the statement of Theorem GG2, since the different choices of $(N_r^{(i)})_{r \in R}$ correspond to the various shuffles of the permutations $(\hat{\pi}_r^{(i)})_{r \in R}$, each yielding a different π_i . □

If $t = 2$, a somewhat simpler argument may be given. We demonstrate it in the special case $m = t = 2$.

Example: $m = t = 2$.

In this case, by Corollary 5.4, Theorem GG1 and (5.4) (using simplified notation)

$$\frac{F_D(\bar{q})}{F_T(\bar{q})} = \prod_{i=1}^2 \prod_{j=1}^n (1 - q_i^{2j}) \cdot \left\{ \sum_{k=0}^n (q_1 q_2)^k \cdot \frac{\sum_{\pi_1 \in S_k} q_1^{2 \cdot \text{major}(\pi_1)} q_2^{2 \cdot \text{major}(\pi_1^{-1})}}{\prod_{i=1}^2 \prod_{j=1}^k (1 - q_i^{2j})} \right\}.$$

$$\begin{aligned}
& \cdot \left\{ \frac{\sum_{\pi_1 \in S_{n-k}} q_1^{2 \cdot \text{major}(\pi_1)} q_2^{2 \cdot \text{major}(\pi_1^{-1})}}{\prod_{i=1}^2 \prod_{j=1}^{n-k} (1 - q_i^{2j})} \right\} = \\
& = \sum_{k=0}^n q_1^k q_2^k \cdot \prod_{i=1}^2 \begin{bmatrix} n \\ k \end{bmatrix}_{q_i^2} \Sigma_{k,i}
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_{k,i} &= \left(\sum_{\pi_1 \in S_k} q_1^{2 \cdot \text{major}(\pi_1)} q_2^{2 \cdot \text{major}(\pi_1^{-1})} \right) \cdot \left(\sum_{\pi_1 \in S_{n-k}} q_1^{2 \cdot \text{major}(\pi_1)} q_2^{2 \cdot \text{major}(\pi_1^{-1})} \right) = \\
& \sum_{\pi_1 \in S_k \wedge \pi_2 \in S_{n-k}} q_1^{2 \cdot (\text{major}(\pi_1) + \text{major}(\pi_2))} q_2^{2 \cdot (\text{major}(\pi_1^{-1}) + \text{major}(\pi_2^{-1}))}.
\end{aligned}$$

Now, observe that every signed permutation in $B_n = C_2 \wr S_n$ may be constructed in the following way:

Step 1. Choose a number $0 \leq k \leq n$.

Step 2. Choose k digits from the set $\{1, \dots, n\}$ and mark them “negative” (the rest will be “positive”).

Step 3. Choose two permutations: $\pi_1 \in S_k$ (on the “negative” digits), and $\pi_2 \in S_{n-k}$ (on the “positive” ones).

Step 4. Choose a shuffle of π_1 and π_2 .

It should be noted that the major index (in the sense defined at the beginning of Section 3) of the resulting signed permutation is independent of the choice in Step 2. In fact, it is exactly the major index of the shuffle (chosen in Step 4) of π_1 and π_2 . On the other hand, the major index of the inverse of the resulting signed permutation does not depend on the choice in Step 4. In fact, it is exactly the major index of the shuffle (defined by Step 2) of π_1^{-1} and π_2^{-1} .

Combining these facts with Theorem GG2, it follows that for any choosen integer $0 \leq k \leq n$ and any given pair of permutations $\pi_1 \in S_k$ and $\pi_2 \in S_{n-k}$

$$\prod_{i=1}^2 \begin{bmatrix} n \\ k \end{bmatrix}_{q_i^2} q_1^{2 \cdot (\text{major}(\pi_1) + \text{major}(\pi_2))} q_2^{2 \cdot (\text{major}(\pi_1^{-1}) + \text{major}(\pi_2^{-1}))} =$$

$$= \sum_{\sigma \in B(\pi_1, \pi_2)} q_1^{2 \cdot \text{major}(\sigma)} q_2^{2 \cdot \text{major}(\sigma^{-1})},$$

where $B(\pi_1, \pi_2)$ is the set of all signed permutations constructed by choosing π_1 and π_2 at the third step.

Thus,

$$\begin{aligned} \prod_{i=1}^2 \begin{bmatrix} n \\ k \end{bmatrix}_{q_i^2} \sum_{\pi_1 \in S_k \wedge \pi_2 \in S_{n-k}} q_1^{2 \cdot (\text{major}(\pi_1) + \text{major}(\pi_2))} q_2^{2 \cdot (\text{major}(\pi_1^{-1}) + \text{major}(\pi_2^{-1}))} = \\ = \sum_{\sigma \in B_n(k)} q_1^{2 \cdot \text{major}(\sigma)} q_2^{2 \cdot \text{major}(\sigma^{-1})}, \end{aligned}$$

where $B_n(k) := \{\sigma \in B_n \mid \sigma \text{ has } k \text{ “negative” digits}\}$. We conclude that

$$\begin{aligned} \frac{F_D(\bar{q})}{F_T(\bar{q})} &= \sum_{k=0}^n q_1^k q_2^k \cdot \sum_{\sigma \in B_n(k)} q_1^{2 \cdot \text{major}(\sigma)} q_2^{2 \cdot \text{major}(\sigma^{-1})} = \\ &= \sum_{\sigma \in B_n} q_1^{2 \cdot \text{major}(\sigma) + k(\sigma)} q_2^{2 \cdot \text{major}(\sigma^{-1}) + k(\sigma)}, \end{aligned}$$

where $k(\sigma)$ is the number of “negative” digits in σ . Note that $k(\sigma) = k(\sigma^{-1})$. Theorem 3.1 completes the proof of the desired result (for $m = t = 2$).

6 Final Remarks

1. The flag major index may be defined on dihedral groups in an analogous way (with respect to the Coxeter generators). An exact analogue of Theorem 4.1 may be derived. This will be proved elsewhere.
2. Using Theorem 3.1 it is possible to connect the flag major index with multiplicities of irreducible representations in homogeneous components of the coinvariant algebra of the group $C_m \wr S_n$. These multiplicities were calculated by Stembridge [Stem] and involve major indices of skew standard Young tableaux. For more details see [AR, Section 5].

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